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Integral geometry, twistors and generalised conformal structures

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To Izrael Moiseevich Gelfand who has educated me in integral geometry.

Abstract. *The Penrose twistor theory is very close to integral geometry in the sense of Gelfand both in the way it states its problems and in the way it uses its technical means. Actually, one can consider the Penrose transform as an analogue of the Radon-John transform for $\bar{\partial}$-cohomologies. However, there are other common points, and their close scrutiny is very instructive for both theories. For example, the study of curved twistor manifolds [1] is very similar to integral geometry of the manifold of curves [2 - 4]. In this paper we present a review of several such «boundary» questions. All consideration are made over $\mathbb{C}$.*

1. Integral geometry for lines in $\mathbb{C}P^3$. We start by reminding some simple facts from the integral geometry in projective space (for more details see [5]). Introducing homogeneous coordinates $z = (z_0, z_1, z_2, z_3)$ into $\mathbb{C}P^3$ let us define lines by pairs of points (w, v). We define an integral transformation on sections f of the (smooth) bundle $O(-2) \times O(-2)$ over $\mathbb{C}P^3$, i.e. actually on homogeneous functions $f(z, \bar{z})$ of bi-degree $(-2, -2)$. Let

$$(1) \qquad \hat{f}(w, v) = \int_{\mathbb{C}P^1_\tau} f(\tau_0 w + \tau_1 v)\sigma(\tau) \wedge \overline{\sigma(\tau)},$$

where

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$$\sigma(\tau) = -\tau_0 d\tau_1 + \tau_1 d\tau_0.$$

We use the fact that the integrand can be considered as defined on $\mathbb{C}P^1_\tau$ and that $\hat{f}$ can be interpreted as a section of a linear bundle over the Grassmannian $G_{2,4}$:

$$\hat{f}(gw, gv) = |\det g|^{-2}\hat{f}(w, v), \quad g \in GL(4, C).$$

In that construction one can replace the bundle $O(-2)$ by any bundle $O(-k)$, $k > 2$. The integrand in (1) has then to be augmented by a monomial in $\tau, \bar{\tau}$ of bi-degree $(k-2, k-2)$, e.g. $|\tau_0|^{2(k-2)}$. The invariant point of view requires consideration of all monomials and interprets $\hat{f}$ as a spinor field.

The image of the transformation f is described by a system of differential equations

$$\Delta_{ij}\hat{f} = \frac{\partial^2 \hat{f}}{\partial w_i \partial v_j} - \frac{\partial^2 \hat{f}}{\partial w_j \partial v_i} = 0, \quad \bar{\Delta}_{ij}\hat{f} = 0, \; 0 \leqslant i < j \leqslant 3 \tag{2}$$

The homogenuity condition implies that one can consider only one pair of indices (i, j).

The central role in that theory is played by the Gelfand-Graev-Shapiro operator $\varkappa$ [6]. It takes functions depending on (w, v) into $(1, 0)$-form depending on v with coefficients depending on w:

$$F(w, v) \to \varkappa_w F = \sum_j \frac{\partial F(w, v)}{\partial w_j} dv_j. \tag{3}$$

Equations (2) imply that the form $\varkappa_w \hat{f}$ is ∂-closed, and, respectively, that the $(1, 1)$-form $(\varkappa_w \wedge \bar{\varkappa}_w)\hat{f}$ is closed on $\mathbb{C}^4_v \setminus \{w\}$. We have

$$\int_\gamma (x_w \wedge \bar{\varkappa}_w)\hat{f} = c(\gamma) f(w) \tag{4}$$

for any 2-dimensional (over $\mathbb{R}$) cycle in $\mathbb{C}^4 \setminus \{w\}$. The coefficient $c(\gamma)$ depends only on γ, does not depend on f and, in general, does not vanish. Therefore, the operator $\varkappa$ provides a series of inversion formulas which are different for different choice of a cycle γ. Hence one can view formula (4) as a universal inversion formula for the integral transformation (1). The corresponding formulas for $k > 2$ are written in the same way.

2. The Penrose transform and its inversion. In this section we follow [7, 8]. Following Penrose, consider domain D_+ of positive and negative twistors

$$(5) \qquad H(z) = |z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2 \gtrless 0$$

in $\mathbb{C}P^3$ and their common boundary Σ, the surface of null twistors. Domains D_+ are 1-linearly concave, i.e. they coincide with the union of lines contained in them. The manifold of lines contained in D_+ is biholomorphically equivalent to the tube of the future T_+, and the lines lying on Σ correspond to the Shilov boundary of that domain (compactified Minkowsky space).

Consider the space $C^{(0,1)}(D_+, O(-k))$, $k > 2$, of (0, 1)-forms in D_+ with coefficients in the fibres of the bundle $O(-k)$ (i.e. homogeneous in z of degree $-k$). If a line $\ell(w, v)$ going through the points w, v lies in D_+ then for any $\varphi \in C^{(0,1)}(D_+, O(-k))$ let

$$(6) \qquad \hat{\varphi}(w, v) = \int \varphi|_{z=\tau_0 w + \tau_1 v} \wedge \sigma(\tau)(\tau_0)^{k-2}.$$

Considering only $\bar{\partial}$-closed forms φ ($\varphi \in Z^{(0,1)}(D_+, O(-k))$), one has a holomorphic function $\hat{\varphi}(w, v)$. For $k = 2$ it can be interpreted as a section of a linear bundle over T_+; and for $k > 2$ over the flag manifold $(v, \ell(w, v))$, $\ell \subset D_+$. The kernel of the mapping $\varphi \to \hat{\varphi}$ consists exactly of $\bar{\partial}$-exact forms ($\varphi \in B^{(0,1)}$). The Penrose transform is the induced mapping on the cohomology space $H^{(0,1)}(D_+, O(-k))$. It turns out that the inversion formula for it can also be written in terms of the operator $\varkappa : F \to \Sigma\, \partial/\partial W_j\, dv_j$.

Let $\tilde{D}_+$ be the manifold of pairs (w, v) for which the line $\ell(w, v)$ lies in D_+ and consider the section Γ of the bundle $\tilde{D}_+ \to D_+$.

THEOREM [7, 8]. *For* $\varphi \in Z^{(0,1)}(D_+, O(-k))$ *the form*

$$(7) \qquad (\varkappa\hat{\varphi}|_\Gamma)^{(0,1)} - \varphi$$

is $\bar{\partial}$*-exact.* ■

In (7) one takes a (0, 1)-component of the restriction of the form $\varkappa\hat{\varphi}$ on to section Γ defined on $\tilde{D}_+$. The result is a (0, 1)-form on D_+. Note that $\varkappa\hat{\varphi}$ is a closed form on $\tilde{D}_+$ since, as one can easily check directly from (6), $\hat{\varphi}$ satisfies the system of equations $\{\Delta_{ij}\, \hat{\varphi} = 0\}$.

The result is the inverse Penrose transform. By taking different sections Γ one has different representatives of a cohomology class of the form φ. In our view formula (7) shows a very important fact: in a sense one obtains a holomorphic continuation of one-dimensional $\bar{\partial}$-cohomologies in the form of holomorphic forms on the Stein manifold $\tilde{D}_+$ with fibres over D_+. Non-holomorphic property φ is connected only with the non-holomorphic property of section Γ. Holomorphic form $\varkappa\,\hat{\varphi}$ provides a very full characterisation of the cohomology class φ and in a number of problems (continuation, uniqueness, etc.) it makes it possible to apply the tools of holomorphic analysis to the study of higher $\bar{\partial}$-cohomologies.

A similar inversion formula is also valid for the inverse Penrose transform defined on $\bar{\partial}_b$-cohomologies on the boundary Σ. Note that for that formula a specific form of the domain D_+ is not important, the only fact is that it is 1-linearly concave (see [7]), i.e. that they coincide with unions of lines contained in them.

3. **Some explicit formulas.** We give two explicit formulas for the Penrose transform inspired by integral geometry. We start by the formula which reconstructs the Penrose transform $\hat{\varphi}$ by the boundary values of φ on $\Sigma = \partial D_+$. We give the formula for $k = 3$ when it especially simple. Let $\varphi \in Z^{(0,1)}(D_+, O(-3))$. Then

$$\hat{\varphi}(w, v) = \frac{1}{(2\pi i)^2 3!} \int_{\Gamma(\Sigma)} \frac{\varphi \wedge [z, ds, dz, dz] \wedge [u, z, v, du]}{[w, z, u, v]^2} \tag{8}$$

Here $[a_1, a_2, a_3, a_4]$ denotes the determinant with columns a_j, some of which may be 1-forms. Computing a determinant one takes skew products of them, so that a non-zero determinant may have identical columns of 1-forms. Now we consider a bundle $\tilde{\Sigma} \to \Sigma$ over the boundary Σ where $\tilde{\Sigma}$ consists of such pairs (z, u), $z \in \Sigma$, $u \in \bar{D}_-$ for which the line $\ell(z, w)$ belongs to $\bar{D}_-$ (lies outside D_+). The integral can be taken over any section $\Gamma(\Sigma)$ of that bundle (for different sections the integrated forms differ by an exact one which does not affect the value of the integral). Consider the expression $[w, z, u, v]$ in the denominator. It vanishes if and only if the lines $\ell(z, u)$ and $\ell(w, v)$ intersect. (This is an analogue of the Cauchy kernel for a pair of lines). For $k > 3$ there is a similar but more complicated formula [9]. The reason is that one has to consider a Leray residual class for a polar singularity of higher order and to fix a representative of the class one has to introduce an additional structure (for $k = 3$ there is a pole of order 1 and a canonical residual form).

Starting with the formula (8) one can obtain a formula reconstructing the form $\varkappa\,\hat{\varphi}$ by the boundary values of φ:

$$x\hat{\varphi} = \frac{1}{(2\pi i)^3 3} \int_{\Gamma(\Sigma)} \frac{\varphi \wedge [z, dz, dz, dz] \wedge [u, z, v, du] \wedge [v, z, u, dv]}{[w, z, u, v]^3} \tag{9}$$

That formula can be naturally interpreted as a formula that reconstructs the cohomology class from $H^{(0,1)}(D_+, O(-3))$ from boundary values. To get the form cohomological to φ it is sufficient to restrict the right-hand side in v to a section $\Gamma(D_+)$ of the bundle $\widetilde{D}_+ \to D_+$ and take the (0, 1)-component. An essential thing, in our point of view, is that the formula (9) does not reconstruct the cohomology class but its holomorphic «continuation» to $\widetilde{D}_+$. For that reason one can work with integral formula with holomorphic kernel which is very convenient in analytical considerations.

Complex integral formulas are closely connected with integral geometry. Thus, the Cauchy-Fantappié-Leray formula is a precise analouge of the inversion formula for the Radon transform (see discussion in [9]). Formulas of this section have also their analouges in real integral geometry [9].

4. Integral geometry for 4-parameter families of curves on 3-dimensional manifolds. In the curved version of the Penrose twistor theory [1] an important role is played by such 4-parameter families of curves E_ξ, $\xi \in \Xi$ on a 3-dimensional manifold X that the intersection relation of curves induces a conformal metrix on Ξ (which is automatically self dual). More precisely, to $x \in X$ one assigns two-dimensional submanifolds $S_x \subset \Xi : \xi \in S_x \Longleftrightarrow x \in E_\xi$. Tangent planes α_x to S_x form a cone $V_\xi \subset T_\xi\, \Xi$. It is required that V_ξ should be quadratic. Then besides the family of α-planes on V_ξ tangent to S_x there is one more family of planes called β-planes. Each of those families is parametrised by points of a projective line. Parametrisation of the family of α-planes makes it possible to continue the curves E_ξ canonically to global rational curves with the normal bundle $O(1) \oplus O(1)$. Conversely, for such a family of rational curves the cones V_ξ are quadratic.

It turns out that the same families of curves play an exceptional role in integral geometry. It is precisely those families of curves for which there exist a local inversion formula of the form (4). A convenient way is to rewrite formula for lines (section 1) in affine coordinates in such a way that it preserves projective invariance.

Consider the set of lines $z = \alpha t + \beta$, z, α, $\beta \in \mathbb{C}^n$, $t \in \mathbb{C}^1$, in $\mathbb{C}^n$ and let

$$\hat{f}(\alpha, \beta) = \int f(\alpha t + \beta)\, dt \wedge d\bar{t}. \tag{10}$$

The operator

$$\varkappa_w : F(\alpha, \beta) \to \Sigma \left(\frac{\partial F}{\partial \beta_j} + c\, \frac{\partial F}{\partial \alpha_j}\right) d\alpha_j + F\, dc \,|_{\beta = w} \tag{11}$$

taking functions depending on (α, β) into (1, 0)-forms on the manifold of pairs

(α, c), $c \in \mathbb{C}^1$ takes each function of the form $\hat{f}(\alpha, \beta)$ into ∂-closed forms. Here α are coordinates on the manifold of lines with $\beta = w$ and c is an additional variable which in the final formula may be considered as arbitrarily depending on α. Formula (11) involves differentiation along a family of lines going through the point on the line $z = \alpha t + w$ for the parameter value $t = -1/c$ where the point c can be chosen independently for different α. As a result $(\varkappa_w \wedge \bar{\varkappa}_w)\hat{f}$ is a closed form and one has the inversion formula (4).

Let now the cones V_ξ on the manifold of curves be quadratic, let w be a fixed point in X and S_w be the corresponding 2-dimensional submanifold in Ξ. Let $\xi \in S_w$ and denote by α_0 the α-plane on V_ξ corresponding to w (the tangent plane to S_w). To each point $c \in E_\xi$ there corresponds an α-plane $\alpha_c \subset V_\xi$. Let L_c be a linear operator from α_0 into α_c which leaves each vector in its β-plane. Such an operator is defined up to a scalar multiple. Let

$$\varkappa_w F(\xi; \tau) = dF(\xi; L_c\tau) + F\theta, \qquad \tau \in \alpha_0, \tag{12}$$

where the value of the form $\varkappa_w F$ on a vector τ is the sum of the value of the differential on the vector $L_c\ \tau \in \alpha_c$ and of the constant form θ multiplied by F. We have given the structure of operator $\varkappa_w$ but it remains to specify the measure with respect to which integrals over curves in E_ξ are taken. There is a canonical projective structure on E_ξ. Fix an affinisation by specifying a submanifold Λ of «infinite points» E_ξ on X, dim $\Lambda = 2$. Then the affine measure is defined up to a scalar multiple which can be normalised by choosing an affine parameter t on E_ξ in such a way that in a neighbourhood of w on E_ξ one has $z = \alpha t + \beta t + o(t)$. For another point w one has a coefficient which is a function depending on ξ. Thus, $\hat{f}(\xi) = \int_{E_\xi} f\, \overline{dt} \wedge dt$ for all $f \in C_0(X)$. Then for an appropriate (1, 0)-form θ the form

$$x_w \hat{f} \quad \text{on} \quad S_w \times \mathbb{C}^1_c$$

is ∂-closed. The form θ can easily be described explicitly but that will not be considered here. The proof is based entirely on the fact that a conformal metrix is equivalent to a flat one in a neighbourhood of each point up to the third order. Hence for each family of rational curves with the bundle $O(1) \oplus O(1)$ there is an inversion formula of the form (4).

Now a few words about the necessity of that condition. In [10] a general problem of the existence of a local inversion formula for the problem of integral geometry on curves is studied. It is assumed that all curves of Ξ are parametrised and that densities $\psi(t; \xi)$ are defined on then so that for each $f \in C_0^\infty(X)$ the integrals

$$\hat{f}(\xi) = \int_E f \,|\, \psi(t;\xi)\,|^2\, dt \wedge \overline{dt}$$

are defined. We investigate when there exists a differential operator of the first order $\varkappa_w$ acting from the set of functions on Ξ into (1, 0)-forms on S_w for which the form $\varkappa_w f$ is ∂-closed. The structure of such operators can be completely described. For dimensions considered here one of the main results can be reformulated in the following way. The operator $\varkappa_w$ is of the form (12) where L_c is replaced by a linear operator L acting from α_0 into $T_\xi\Xi$ such that for each $\tau \in \alpha_0$ the plane τ, $L\tau$ intersects all α-planes. It means that that plane belongs entirely to V_ξ and that each vector $\tau \in \alpha_0$ is contained in a plane on V_ξ which does not belong to a family of α-planes (i.e. a β-plane). Thus there exist two families of 2-dimensional planes on V_ξ which implies that the cone V_ξ is quadratic and that L maps α_0 on an α-plane. Paper [10] also studies conditions the family of densities has to satisfy under which the form $(\varkappa_w \wedge \bar{\varkappa}_w)\hat{f}$ is closed.

5. Integral geometry for manifolds of rational curves. As we have already noted, the paper [10] studies the structure of the operator $\varkappa_w$ which provides an inversion formula for any family of curves (with any number of parameters and on a manifold of any dimension). With the use of that description the paper [2] shows that *any manifold of curves for which such an operator exists is necessarily (after an appropriate continuation of curves) a full family of rational curves*. It is full in the following sense.

Consider the space of sections of the normal bundle to the curve E_ξ (infinitesimal deformations) which are continued to local deformations of the curve. That space is required to coincide with the set of all sections of a vector bundle over $E_\xi \cong \mathbb{C}P^1$. The central point of the proof is the fact that the intersection relation on full manifolds of rational curves induces a conformal structure of high order i.e. a generalisation of the conformal 4-metrix. We now describe those structures. A generalised conformal structure is defined by a cone. To manifolds of rational curves there correspond cones $V_{(k_1,\ldots,k)} \subset \mathbb{C}^n$ defined by ℓ-tuples of integers $k = (k_1, \ldots, k_\ell)$, $k_1 \geqslant \ldots \geqslant k_\ell \geqslant 0$, $\Sigma\, k_j = n - \ell$. The cone $V_{(k)}$ in the coordinate space (x^i_j), $1 \leqslant i \leqslant \ell$, $0 \leqslant j \leqslant k_i$, is a union of planes $\alpha(\tau)$, $\tau = (\tau_0, \tau_1)$ of codimension ℓ:

$$\begin{aligned} &\tau_0^{k_1} x_0^1 + \tau_0^{k_1 - 1}\tau_1 x_1^1 + \ldots + \tau_1^{k_1} x_{k_1}^1 = 0, \\ (13) \qquad &\ldots \\ &\tau_0^{k_\ell} x_0^\ell + \tau_0^{k_\ell - 1}\tau_1 x_1^\ell + \ldots + \tau_1^{k_\ell} x_{k_\ell}^\ell = 0. \end{aligned}$$

where the coefficient before the coordinate x^i_j equals $\tau_0^{k_i - j}\,\tau_1^j$. We have the family of planes parametrised by points of projective line $\mathbb{C}P^1$. Its union is denoted by $V_{(k)}$. A remarkable fact is that besides the family of α-planes there is one more family of 2-dimensional planes $\beta(u)$ on $V_{(k)}$ parametrised by points of the projective space $\mathbb{C}P^{n-\ell}_u$. Denote the homogenous coordinates u by u^i_j where, as above, $1 \leqslant i \leqslant \ell$ but $0 \leqslant j \leqslant k_i - 1$. Then $\beta(u)$ consists of the points x with the coordinates

$$x^i_j(\tau) = \tau_0 u^i_{j-1} - \tau_1 u^i_j, \; x^i_j(\tau) \in \alpha(\tau), \tag{14}$$

where (τ_0, τ_1) are coordinates on $\beta(u)$. Under some natural assumptions $V_{(k)}$ is a unique family of cones containing two families of planes one of which is two-dimensional.

Let us say that a $\mathscr{P}_{(k)}$-structure is given on a manifold Ξ, dim $\Xi = n$, if in each tangent space a cone V_ξ linearly equivalent to V_k is specified. That structure is called integrable if the distribution of α-planes is integrable, i.e. if there exists such a family of submanifolds S_z that all the tangent planes to S_z are α-planes and each α-planes is tangent to some S_z. The manifold X of parameters z (evidently, dim $X = \ell + 1$) is called the twistor manifold for an integrable $\mathscr{P}_k$-structure and the points of Ξ correspond to curves E_ξ on X. For $n = 4$, $k_1 = k_2 = 1$ one has the Penrose construction.

The projective structure on the family of α-planes in $V_\xi \cong V_{(k)}$ induces locally the structure of a rational curve on the curves of E If $k_2 > 0$ on has a full family of rational curves (globally).

For $k_2 = \ldots = k_\ell = 0$ the situation is more complicated. A $\mathscr{P}_{(k)}$-structure is flat if $\Xi = \mathbb{C}^n$, and the cones $V_\xi \cong V_{(k)}$ are obtained one from another by a shift. In that case Ξ is realised as a space of all sections of the bundle $O(k_1) \oplus \ldots \oplus O(k_\ell)$ on $\mathbb{C}P^1$. If $k_2 > 0$, then every $\mathscr{P}_{(k)}$-structure in a neighbourhood of a point is equivalent up to the 3d order to a flat structure. That statement may be considered as an analogue of the classical Desargues theorem of projective geometry. For $k_2 = \ldots = k_\ell = 0$ that condition has to be imposed and then one has a full family of rational curves on the twistor manifold X. The Desargues condition can be elegantly written in the analytical form [3]. The situation is similar to that in the projective geometry: on the projective plane the Desargues condition is taken for an axiom while in the space of higher dimension it is proved.

Conversely, let Ξ be a full manifold of rational curves E_ξ on X, dim $X = \ell + 1$. Consider a generalised conformal structure on Ξ induced by the intersection relation of curves. Then the cones V_ξ arising in the tangent spaces are linearly equivalent to the cone $V_{(k)}$ for some $(k_1, \ldots, k_\ell)$. It is an immediate consequence of the Grothendieck theorem describing the structure of vector bundles over a projective line. The reason is that vectors of the tangent space $T_\xi \Xi$ correspond to sections of the normal bundle over $E_\xi \cong \mathbb{C}P^1$

For integral geometry on families of rational curves all the considerations repeat the reasoning of section 4: the operator $\varkappa_w$ is of the form (12) since we have β-planes, and one takes affine densities on E_ξ corresponding to the canonical projective structure on them. The guidelines of a converse reasoning: the description of admissible operators x in [10] implies the necessity of the existance of 2-dimensional β-planes on V_ξ, and, consequently, of a $\mathcal{P}_{(k)}$-structure on E_ξ.

Now a few words about the relation to non-linear differential equations. If a $\mathcal{P}_{(k)}$-structure is given on the manifold Ξ then the distribution of α-planes is given by the system of linear differential equations of the first order with the rational parameter τ:

$$
(16)\qquad
\begin{aligned}
&\tau_0^{k_1}\omega_0^1 + \ldots + \tau_1^{k_1}\,\omega_{k_1}^1 = 0\\
&\ldots \qquad \ldots \qquad \ldots\\
&\tau_0^{k_\ell}\omega_0^\ell + \ldots + \tau_1^{k_\ell}\,\omega_k^\ell = 0,
\end{aligned}
$$

where $\{\omega_j^i\}$ is a full family of (1, 0)-forms. The integrability condition for that system is a non-linear differential equation on its coefficients. The ideology of the inverse scattering problem approach is that many non-linear differential equations admit such a representation (as an (L, A)-pair): as a compatibility condition for a system of linear differential equations with rational (spectral) parameter τ. For example, for $n = 4$, $k_1 = k_2 = 1$ we have the equation of self-duality for a conformal 4-metrix.

Solutions of such a non-linear system correspond to integrable linear systems with a parameter. The twistor ideology in that situation can be explained in the following way: to each integrable system (16) there corresponds a full system of rational curves E_ξ on the twistor manifold X. Thus, instead of constructing integrable systems (16) we can construct families of rational curves. It is, therefore, important to develop a technique for constructing a sufficient supply of such families.

6. Reductions of manifolds of rational curves. For a self-dual metrix Penrose has proposed [1] the following construction which can be used for any k. Consider a «flat» family: the set of sections of the vector bundle $O(k_1) \oplus \ldots \oplus O(k_\ell)$ on $\mathbb{C}P^1$. It turns out that if one considers a perturbed complex structure in the total space of the bundle, then the perturbed manifold will also have a full family of rational curves with the same normal bundle.

In geometry new structures are usually obtained either by perturbing a flat one or by restricting multidimensional (e.g. flat) structures to submanifolds of lesser dimension.

Evidently, a restriction of a $\mathscr{P}_{(k)}$-structure to a submanifold is no longer a $\mathscr{P}_{(k)}$-structure. It turns out, however, that one can fully describe those submanifolds on which there arises an induced $\mathscr{P}_{(k)}$-structure [2, 3]. Let Ξ be a manifold with an integrable $\mathscr{P}_{(k)}$-structure. Then can be realised as a full system of rational curves E_ξ on some manifold X. We are interested in the question on which submanifolds $\Pi \subset \Xi$ there is an induced $\mathscr{P}_{(k)}$-structure, or, equivalently, which subfamilies of rational curves $\Pi \subset \Xi$ are full,. We give two ways of specifying a subfamily of curves:

(i) fix a submanifold $\Gamma \subset X$, codim $\Gamma > 1$ and consider the subfamily $\Xi(\Gamma) \subset \Xi$ of curves intersecting Γ;

(ii) fix a submanifold $S \subset X$, codim $\Gamma = 1$, and consider a subfamily $\Xi(S)$ of all curves E_ξ tangent to S.

THEOREM. *Any subfamily of curves Π in a full family Ξ of rational curves E_ξ defined by a set of conditions of the form (i), (ii) is full (i.e. there is an induced $\mathscr{P}_{(k)}$-structure on Π). Any full subfamily in a general position can be defined by a set of such conditions (*).*

In fact one can fully describe full subfamilies $\Pi \subset \Xi$ of rational curves without assuming that they are in general position. Such a description uses the language of σ-processes [2, 3].

The above theorem makes it possible to construct a large number of examples. For example, it automatically solves the problems: for which families of lines or conical sections the problem of integral geometry has a local solution, for what 4-parameter families of conical sections in $\mathbb{C}P^3$ the intersection relation induces a conformal self-dual 4-metrix. In the latter one has to impose 4 conditions of intersection with a fixed curve Γ, or of being tangent to a fixed surface S (in general position there are no other examples).

Now a few words about the proof. We are interested in such submanifolds $\Pi \subset \Xi$ for which $T_\xi \Pi \cap V_\xi$ is again a cone of the family $V_{(k)}$. The integrability condition is, evidently, again valid. Therefore, the first task is to find those planes (it is sufficient to find hyperplanes) which in the intersection whith $V_{(k)}$ give a cone of the same class. This is a linear algebra problem and it turns out that a hyperplane has to contain an α-plane (which is a necessary and sufficient

(*) Notice that the dimensionality of the twistor manifold X does not decrease in the course of this procedure. In contrast to the Penrose constructiokn the complex structure on X is not changed but some birantional transformation is performed by the way.

condition). One obtains an algebraic system of equations on the parameters of a hyperplane which results in an explicit system of non-linear differential equations on Π. It turns out that the system can be integrated exactly by a generalisation of the Hamilton-Jacobi method, and the result is given by the above theorem. Points of submanifolds Γ, S correspond to generalised bi-characteristics which are combined into a solution by a rather complicated glueing together procedure (formulated, in general, in the σ-processes langauge). An interesting fact is that it is more convenient to study the differential equation on a full subfamily of rational curves in the language of the manifold Ξ, but the result is more conveniently formulated in the twistor language.

In our view, the class of $\mathscr{P}_{(k)}$-structures is very interesting and deserves a more close study. Here we have discussed only restrictions of these structures. Another interesting question is that of embedding of manifolds having those structures into manifolds of higher dimension with a flat $\mathscr{P}_{(k)}$-structure. An interesting observation has been made by A.B. Goncharov (private communication): projectivisations $U_{(k)}$ of the cones $V^*_{(k)}$, dual to the cones $V_{(k)}$ are algebraic varieties of minimal degree (they have the minimal possible degree among algebraic varieties of given dimension in $\mathbb{C}P^{n-1}$). The other examples of varieties of minimal degree are quadrics (they correspond to usual conformal structures). The only other example is given by the Veronese surface in $\mathbb{C}P^5$ and the cone over it (by the Enriques theorem). Thus usual conformal structures and $\mathscr{P}_{(k)}$-structures have an exceptional position among generalised conformal structures.

7. **Generalised metric structures associated to $\mathscr{P}_{(k)}$-structures.** A problem of constructing an integrable $\mathscr{P}_{(k)}$-structure (i.e. a full manifold of rational curves) is often a part of some problem of mathematical physics. Thus, the problem about self-dual conformal 4-metrices is only a part of the problem about right flat 4-metrices (self-dual solutions of the vacuum Einstein equation). It is then important to specify a geometric structure corresponding to the full problem which is an extension of the $\mathscr{P}_{(k)}$-structure. Such an extension will be called a generalised metric structure. An extension of any individual structure can be effected in different ways. If our aim is to construct solutions by restricting multidimensional structures, one has to consider a compatible extension of the whole series of generalised conformal structures.

For example, a 4-dimensional metrix can, of course, be extended to a complex 4-metrix on Ξ but that extension is not continued to other $\mathscr{P}_{(k)}$-structures. In [2, 11] another extension in the form of a bundle of 2-forms is proposed. Thus, a self-dual 4-metrix corresponds to an integrable system

$$\omega^1(\tau) = \tau_0 \omega_0^1 + \tau_1 \omega_1^1 = 0 \tag{17}$$
$$\omega^2(\tau) = \tau_0 \omega_0^2 + \tau_1 \omega^2{}_1 = 0.$$

Consider a quadratic bundle of 2-form

$$F(\tau) = \omega^1(\tau) \wedge \omega^2(\tau). \tag{18}$$

The condition

$$dF(\tau) = 0 \tag{19}$$

for all τ is a stronger one that the integrability condition of the system (17) for all τ and it ensures that the metrix

$$g = \omega_0^1\, \omega_1^2 - \omega_1^1\, \omega_0^2 \tag{20}$$

satisfies the vacuum Einstein equation. Thus, the self-dual Einstein equation is equivalent to the following system of conditions on a quadratic bundle of 2-forms:

$$\begin{aligned} &\text{(i)} && F(\tau) \wedge F(\tau) = 0 \\ &\text{(ii)} && F(\tau) \wedge F(\sigma) \neq 0 \quad \text{if} \quad \tau \neq \lambda\sigma \\ &\text{(iii)} && dF(\tau) = 0. \end{aligned} \tag{21}$$

Condition (i) ensures that the bundle can be represented in the form (18) and condition (ii) implies completeness of the system of the forms ω_j^i (non-degeneracy of the metrix).

Despite the fact that intestigation of the bundle $F(\tau)$ is equivalent to the study of the metrix g, it is another geometrical structure. For the metrix the gauge group is $SO(4) \cong SL(2) \times SL(2)$. A right flat metrix, in a natural sense is flat with respect to one of the factors $SL(2)$. Going from g to $F(\tau)$ we reduce the gauge group $SO(4)$ to that factor: for $F(\tau)$ the gauge transformations include only projective transformation of the parameter (τ_0, τ_1).

Consider now similar multidimensional constructions. Note that the description of the induced $\mathscr{P}_{(k)}$-structures given in the preceding section does not affect $\mathfrak{L}$, i.e. to obtain a 4-metrix one has to consider on Ξ, $\dim \Xi = k_1 + k_2 + 2$ system of the form

$$\omega^1(\tau) = \tau_0^{k_1} \omega_0^1 + \ldots + \tau_1^{k_1} \omega_{k_1}^1 = 0,$$
$$\omega^2(\tau) = \tau_0^{k_2} \omega_0^2 + \ldots + \tau_1^{k_2} \omega_{k_2}^2 = 0.$$

Taking $F(\tau)$ defined by formula (18) let us require that it satisfies (20). Then the $\mathscr{P}_{(k_1,\, k_2)}$-structure is integrable and the corresponding bundle of 2-forms

$F(\tau)$ of degree $k_1 + k_2$ will be called a corresponding metric structure. Then $F(\tau)$ satisfies (i), (ii) from (21).

Let now a $\mathscr{P}_{(1,1)}$-structure (conformal metrix) be induced on $\Pi \subset \Sigma$, dim $\Pi = 4$. Restricting $F(\tau)$ to Π one obtains a bundle of 2-forms satisfying (21), however its degree in τ equals $k_1 + k_2$. The fact that we have obtained a $\mathscr{P}_{(1,1)}$-structure on Π implies that

$$F(\tau) = \psi(\tau, \xi)\widetilde{F}(\tau) \tag{23}$$

where $\widetilde{F}(\tau)$ is a quadratic bundle of 2-forms on Π and the function $\psi(\tau, \xi)$, $\xi \in \Pi$ is a homogeneous polynomial in τ of degree $k_1 + k_2$. Then $\widetilde{F}$ satisfies conditions (i), (ii) in (21), but, in general, does not satisfy (19). Note that the function ψ is defined up to a factor depending on $\xi \in \Pi$. One has to impose special conditions on submanifolds Γ_i, S_j which have to be tangent to / intersect curves from Π in order for the form $F(\tau)$ to be closed for some ψ.

Suppose that a system (22) corresponds to a flat structure. Then $\omega_j^i = \partial\xi_j^i$ where $\{\xi_j^i\}$ are coordinates on Ξ and E_ξ is of the form

$$z^1(\tau) = \xi_0^1\,\tau_0^{k_1} + \ldots + \xi_{k_1}^1\,\tau_1^{k_1}$$
$$z^2(\tau) = \xi_0^2\,\tau_0^{k_2} + \ldots + \xi_{k_2}^2\,\tau_1^{k_2},$$

where $(z^1, z^2, \tau_0, \tau_1)$ are homogeneous coordinates on the twistor manifold X which is the total space of the bundle $O(k_1) \oplus O(k_2)$ on $\mathbb{C}P^1$. We have to impose $k_1 + k_2 - 2$ tangency-intersection conditions. Let us show two situations when one can preserve the fact that $\widetilde{F}(\tau) = dz^1(\tau) \wedge dz^2(\tau)$ is closed. First, if each of the curves $\Gamma_1, \ldots, \Gamma_{k_1+k_2-2}$ in X lies over the same point $\mathbb{C}P^1$ (i.e. for $a\tau_0 - b\tau_1 = 0$ with constant a, b) then ψ depends only on τ and does not depend on $\xi \in \Xi$ and hence the fact that $F(\tau)$ is closed is inherited by the forms $\widetilde{F}(\tau)$. That fact alone already provides some interesting examples [2, 4, 12]. However, those solutions have singularities.

The second construction lends to non-singular solutions. It involves the tangency conditions for submanifolds S_j satisfying some strong global compatibility conditions. Adding one more variable w, let $P(w, z^1, z^2, \tau_0, \tau_1) = w^\nu + \ldots$ be a polynomial of weighted degree $m\nu$ where one ascribes the weights m, k_1, k_2, 1, 1 to the variables $w, z^1, z^2, \tau_0, \tau_1$ respectively. Let Π be a submanifold of curves of the form (24) in Ξ which can be lifted to submanifold $\Sigma_p = \{P = 0\}$ and let $w(\tau)$ be a fixed lifting (*). Consider a function

(*) Those are tangency conditions for projections of the branching manifolds Σ_p to X.

$$D(w, z, \tau) = \prod_{j=1}^{\nu-1} (w - w_j)$$

on Σ_P where the product is taken over all roots w_j of the polynomial P conjugate to w. Let now $m(\nu - 1) = k_1 + k_2 - 2$ and suppose that Π is non-empty. Then

$$\tilde{F}(\tau) = F(\tau)|_{\Pi} \,/\, D(w(\tau), z^1(\tau), z^2(\tau), \tau) \tag{25}$$

is a closed form. Condition (25) imposes very strong restrictions on the degree of P: $k_1 + k_2 \leqslant deg\ P \leqslant 2k_1 + 2k_2 - 4$. However, some possibilities remain: for $k_1 = k_2$, $m = 2$ there are solutions constructed in [13]. Other examples can also be considered. It would be interesting to investigate what set of solutions can be supplied by such a construction. Can one obtain in that way all asymptotically locally Euclidean solutions of the self-dual Einstein esuation?

Those considerations may be generalised to the problem of constructing Hyper-Kahler metrices. For the corresponding $\mathscr{P}_{(k)}$-structure $\ell = 2m$, $k_1 = \ldots = k_{2m} = 1$, and the generalised metric structure is given by a bundle of 2-forms

$$F(\tau) = \omega^1(\tau) \wedge \omega^2(\tau) + \ldots + \omega^{2n-1}(\tau) \wedge \omega^{2m}(\tau),$$

$$\omega^j(\tau) = \tau_0 \omega_0^j + \tau_1 \omega_1^j.$$

If $dF(\tau) = 0$ then

$$g = (\omega_0^1 \omega_1^2 - \omega_1^1 \omega_0^2) + \ldots + (\omega_0^{2m-1} \omega_1^{2m} - \omega_1^{2m-1} \omega_0^{2m})$$

is a Hyper-Kahler metrix. A bundle of forms is introduced in a similar fashion if $\omega^j(\tau) = \tau_0^{k_j} \omega_0^{k_j} + \ldots + \tau_1^{k_j} \omega_{k_j}^j$.

The problem of restricting those structures to submanifolds is considered as above for $m = 1$ (right flat metrix).

Similarly to the case $m = 1$ one can construct examples of metrices by considering curves Γ_j lying over fixed points of $\mathbb{C}P^1_\tau$. However, we have unable to obtain an analogue of the second construction involving lifting of the curves for $m > 1$.

8. Generalised conformal structures associated to the problems of integral geometry for submanifolds of dimension greater than 1. The case of submanifolds of dimension greater that one is considered in integral geometry only in less general situations. In those problems it is natural to follow some basic examples. We remind the reader that integral geometry in that case stems from the following observation. Derivation of the Plancherel formula for the group $SL(2, \mathbb{C})$ in [16] is entirely based on reconstructing a function f in $\mathbb{C}^3$ from its integrals over lines intersecting a hyperbola. An explicit local formula has been found solving that problem. Later it turned out that one can replace the hyperbola by an arbitrary curve, and the local formula inversion still exists. Then all manifolds of lines in general position have been found for which such a formula exists and the role of tangency-intersection conditions in their description has been clarified. That line of research has been completed by describing the general form of operator x on full manifolds of rational curves. It turned out that many facts of the harmonic analysis on SL(2; $\mathbb{C}$) may be transferred to such a non-homogeneous situation.

In [17] it has been shown that the problem about the Plancherel formula for any complex semi-simple Lie groups is also reduced to a problem of integral geometry. For each such group G one considers the manifold of orispheres, i.e. two-sided shifts of a maximal unipotent subgroup. The problem is to reconstruct a function on the group if its integrals over orispheres are given. I.M. Gelfand has repeatedly formulated the problem of finding an inversion formula for some families of submanifolds, including, in particular, families of orispheres on complex semi-simple Lie groups. In the paper [5] which we have already cited above, a differential operator $\varkappa_w$ is constructred taking an integral $\hat{f}$ over planes of dimension p into a closed form on the manifold of planes going through the point w. For $p = 1$ it is described in section 1. In the general case one obtains different inversion formulas by integrating that form $\varkappa_w f$ over different cycles. For the group $SL(n, \mathbb{C})$ orispheres can be interpreted as planes and the operator $\varkappa_w$ provides an inversion formula for the family of orispheres.

In the case of other groups orispheres are in fact curved. Recently, the author has been able [18] to construct operator $\varkappa$ for curved submanifolds which made it possible in particular, to derive an inversion formula for orispheres on any group from the general results of integral geometry.

At the same time one obtains inversion formulas for families of submanifolds that are not related to groups. The main point here is the existance of a remarkable generalised conformal structure on the manifold of orispheres for any semi-simple Lie group. As in section 4 we construct incidence cones $V_\xi \subset T_\xi \Xi$ on the manifold of orispheres Ξ by considering submanifolds of orispheres $S_g \subset \Xi$ going through $g \in G$ and by taking for each $\xi \in \Xi$ the union of tangent planes σ_g to S_g for $S_g \ni \xi$. Those cones turn out to have a very simple structure which is in many respects general for all groups.

Namely, there is a family Σ_1 of one-dimensional subspaces on V_ξ lying in the same two-dimensional subspace; for each line from Σ_1 there is a one-parameter family of two-dimensional subspaces Σ_2 going through it and lying in the same 3-dimensional subspace etc. For each k-dimensional subspace from the family Σ_k there is a one-dimensional family of $(k + 1)$-dimensional subspace containing it and lying in a $(k + 2)$-dimensional subspace, and so on up to dimension S_g. The union of those subspace gives the cone V_ξ.

A further analysis shows that if on a family Ξ of submanifolds $E_\xi \subset X$ the incidence cones V_ξ are of that form and if the corresponding conformal structure is equivalent to a flat one up to the third order, then one can define the operator $\varkappa$ giving an inversion formula. It would be interesting to investigate those structures and the plane of those among them that correspond to groups. The inductive character of V_ξ is a geometric expression of the group root structure.

Summarizing, one can say that an important role in integral geometry is played by generalised conformal structures. In the integrable case one has an incidence relation between Ξ the manifold of submanifolds E_ξ, and the (twistor) manifold X in which E_ξ lie. That incidence relation is conveniently expressed in the language of double fibrations

$$X \swarrow \overset{A}{} \searrow \Xi$$

where A is the manifold of pairs (x, ξ), $x \in E_\xi$. However, we believe that infinitesimal language of generalised conformal structures is in a number of questions more effective. It would be interesting to continue the study of such structures which ensure the existance of local inversion formulas.

It would also be interesting to investigate parallel constructions in the theory of non-linear differential equations which have to be connected with the case of several spectral parameters.

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